

## An oft cited letter from Tate to Serre on computing local heights on elliptic curves

This document contains a letter written by John Tate to Jean-Pierre Serre in 1979 in which Professor Tate describes a fast and elegant algorithm for computing local canonical heights on elliptic curves over local fields  $K_v$  other than  $K_v = \mathbb{C}$ . Accounts of Tate's algorithm were published in [1] and [2], the former including a modified algorithm that also allows  $K_v = \mathbb{C}$ . But over the years there have been numerous articles that make direct reference to the original handwritten letter, so Professor Tate has kindly given me permission to post this typeset version on the ArXiv.

Joseph H. Silverman  
July 24, 2012

- [1] J. H. Silverman, Computing heights on elliptic curves, *Math. Comp.* **51** (1988), 339–358.
- [2] H. Tschöpe and H. Zimmer, Computation of the Néron-Tate height on elliptic curves, *Math. Comp.* **48** (1987), 351–370.

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October 1, 1979

Dear Serre,

Here is an appendix to my letter of 21 June 1968. I want to explain what seems to me an efficient way to compute  $\lambda_\infty$ , the “canonical quasi-function” on an elliptic curve over  $\mathbb{R}$ . The method can be used on an HP25 (see enclosed programs). On the other hand, from a theoretical point of view it can be used to show the existence of the  $\lambda_v$  on any local field  $K_v$  *except* in case  $K_v \cong \mathbb{C}$ . The trick is to use as parameter a function  $t = \frac{1}{x+r}$  with a double zero at 0, but with  $r$  chosen so that the 2 poles of  $t$  are conjugate quadratic over  $K_v$  rather than being in  $K_v$ , so that  $t$  is *bounded* on the group  $E(K_v)$  of rational points, and the computer can compute  $t(2^n P)$  by iteration, *never overflowing*. The method is summed up in:

**Theorem.** *Let  $K$  be a local field. Let  $E$  be an elliptic curve over  $K$  defined by the usual equation*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

*and define  $b_2, b_4, b_6$ , and  $b_8$  as usual. Suppose there is an open subgroup  $\Gamma$  of  $E(K)$  such that  $x(P)$  does not take the value 0 for  $P \in \Gamma$ .*

[Examples: ( $K = \mathbb{R}$ ) If  $b_6 < 0$ , we can take  $\Gamma = E(\mathbb{R})$ ; if  $\Delta > 0$ , and  $b_2 = 0$  (or more generally if the point of order 2 on the connected

component of 0 has a strictly positive  $x$ -coordinate) then we can take  $\Gamma = E^0(\mathbb{R})$ , the connected component of 0 in  $E(\mathbb{R})$ .]

Suppose  $P \in \Gamma$ . Define sequences  $(t_n)$ ,  $(W_n)$ , and  $(Z_n)$ ,  $n \geq 0$ , inductively by:

$$t_0 = \frac{1}{x(P)} \text{ and } t_{n+1} = \frac{W_n}{Z_n}, \text{ where } \begin{cases} W_n = 4t_n + b_2t_n^2 + 2b_4t_n^3 + b_6t_n^4, \\ Z_n = 1 - b_4t_n^2 - 2b_6t_n^3 - b_8t_n^4. \end{cases}$$

There are constants  $\varepsilon > 0$  and  $M$ , independent of  $P$ , such that

$$|t_n| \leq M \quad \text{and} \quad |Z_n| \geq \varepsilon \quad \text{for all } n,$$

and consequently there is a bound  $N$  such that  $\log |Z_n| \leq N$  for all  $n$  and all  $P \in \Gamma$ . Put

$$\mu(P) = \sum_{n=0}^{\infty} \frac{1}{4^n} \log |Z_n| \quad \text{and} \quad \lambda(P) = \frac{1}{2} \log |x(P)| + \frac{1}{8} \mu(P).$$

Then  $\lambda$  is a function continuous on  $\Gamma - \{0\}$ , bounded outside of any neighborhood of 0 in  $\Gamma$ , such that

$$\lambda(P) - \frac{1}{2} \log |x(P)| \rightarrow 0 \quad \text{as} \quad P \rightarrow 0,$$

and such that

$$\lambda(2P) = 4\lambda(P) - \log |Y(P)|, \quad \text{if } 2P \neq 0, \text{ where } Y = 2y + a_1x + a_3.$$

*Proof.* We have  $Y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$ . Let  $x_2 = x(2P)$ . Then

$$(*) \quad x_2 = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6} = \frac{x^4 Z}{Y^2},$$

where

$$Z = 1 - b_4x^{-2} - 2b_6x^{-3} - b_8x^{-4}.$$

Now, under our underlined hypothesis, there is an  $\varepsilon > 0$  such that

$$|x(P)| \geq \varepsilon \quad \text{for} \quad P \in \Gamma.$$

Hence the function  $x^{-1}$  is bounded on  $\Gamma$ , and so  $Z$  is also. But  $Z$  is bounded away from 0, too, on  $\Gamma$ , because  $|x_2| \geq \varepsilon$ , and  $Z$  has no zero in common with  $x^{-1}$  and  $Y$ . Thus  $\log |Z|$  is bounded on  $\Gamma$ . From  $(*)$  we see that, in the notation of the theorem, we have  $Z_n = Z(2^n P)$ , and consequently

$$\mu(P) = \sum_{n=0}^{\infty} \frac{1}{4^n} \log |Z(2^n P)|$$

is well-defined, continuous, and bounded on  $\Gamma$ ; and satisfies

$$(**) \quad \mu(P) = \log |Z(P)| + \frac{1}{4} \mu(2P).$$

From (\*) we find for  $P$  such that  $2P \neq 0$ ,

$$\log|x(2P)| = 4\log|x(P)| + \log|Z(P)| - 2\log|Y(P)|.$$

Eliminating  $\log|Z(P)|$  from these last two equations we find that the function

$$\lambda = \frac{1}{2}\log|x| + \frac{1}{8}\mu$$

satisfies

$$\lambda(2P) = 4\lambda(P) - \log|Y(P)| \quad \text{for } P \in \Gamma, 2P \neq 0.$$

Moreover,

$$\lambda(P) - \frac{1}{2}\log|x(P)| = \frac{1}{8}\mu(P) \rightarrow 0 \quad \text{as } P \rightarrow 0,$$

because  $\mu$  is continuous and  $\mu(0) = 0$ , as one sees by putting  $P = 0$  in (\*\*) and using  $Z(0) = 1$ .  $\square$

Salut et Fraternité

J. Tate